

6.4 - Special Functions ^{nu}

Bessel's equation of order ν : $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$

Example: Find the general solution of the given differential equation on $(0, \infty)$.

$$16x^2 y'' + 16xy' + (16x^2 - 1)y = 0$$

$$x^2 y'' + xy' + (x^2 - \frac{1}{16})y = 0$$

$$\nu^2 = \frac{1}{16} \Rightarrow \nu = \pm \frac{1}{4}$$

$$\text{solution: } y = c_1 J_{1/4}(x) + c_2 J_{-1/4}(x)$$

Bessel Functions of the First and Second Kinds

Assuming a solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ and substituting into Bessel's equation, we have

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \sum_{n=0}^{\infty} \nu^2 c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+r)(n+r) - \nu^2] c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0$$

$$\text{Reindexing yields } \sum_{k=0}^{\infty} [(k+r)^2 - \nu^2] c_k x^{k+r} + \sum_{k=2}^{\infty} c_{k-2} x^{k+r} = 0$$

For $k = 0$ we find $r = \pm \nu$ and $c_1 = 0$.

$$\text{Then } c_k = -\frac{c_{k-2}}{(k+r)^2 - \nu^2}, k = 2, 3, 4, \dots$$

Note: we'll only have nonzero terms if

k is even. Relabel: $z_n = k$ with $r = \nu, -\nu$

$$c_{2n} = - \frac{1}{(z_n + \nu)^2 - \nu^2} c_{2n-2}, \quad n = 1, 2, 3, \dots$$

$$4n^2 + 4n\nu + \nu^2 - \nu^2$$

$$c_{2n} = - \frac{1}{2^2 n(n+\nu)} c_{2n-2}, \quad n = 1, 2, 3, \dots$$

$$n=1 \quad c_2 = - \frac{1}{2^2(1+\nu)} c_0$$

$$n=2 \quad c_4 = - \frac{1}{2^2 \cdot 2(2+\nu)} c_2 = \frac{1}{2^4 \cdot 2(2+\nu)(1+\nu)} c_0$$

$$n=3 \quad c_6 = - \frac{1}{2^6 \cdot 2 \cdot 3(3+\nu)(2+\nu)(1+\nu)} c_0$$

$$\text{In general, } c_{2n} = \frac{(-1)^n}{2^{2n} n! (1+\nu)(2+\nu) \cdots (n+\nu)} c_0$$

A standard choice for c_0 is $c_0 = \frac{1}{2^\nu \Gamma(1+\nu)}$, where $\Gamma(1+\nu)$ is the gamma function. ↙ gamma

Definition: The **gamma function** is the integral-defined function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

Note that $\Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt$

$$u = t^x \quad dv = e^{-t} dt$$

$$du = x t^{x-1} dt \quad v = -e^{-t}$$

$$\Gamma(x+1) = -\cancel{t^x} e^{-t} \Big|_0^{\infty} + x \int_0^{\infty} t^{x-1} e^{-t} dt$$

$$\Gamma(x+1) = x \Gamma(x) \quad (\text{we're using } v)$$

$$\text{So } \Gamma(\underline{v+1}) = \underline{v} \Gamma(\underline{v})$$

$$\Gamma(v+2) = \Gamma(\underline{v+1}+1) = (\underline{v+1}) \Gamma(\underline{v+1})$$

$$\Gamma(v+3) = \Gamma(v+2+1) = (v+2)(v+1) \Gamma(v+1)$$

$\vdots$

$$\Gamma(v+n+1) = (v+n) \cdots (v+2)(v+1) \Gamma(v+1)$$

(like a generalized factorial)

So using a standard choice for c_0 ,

$$c_{2n} = \frac{(-1)^n}{2^{2n+v} n! \Gamma(1+v+n)}, \quad n=1, 2, 3, \dots$$

The function $y = x^\nu \sum_{n=0}^{\infty} c_{2n} x^{2n}$ is then

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu} = J_\nu(x)$$

We also have that $J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 - \nu + n)} \left(\frac{x}{2}\right)^{2n-\nu}$

Definition: The functions $J_\nu(x)$ and $J_{-\nu}(x)$ defined above are **Bessel functions of the first kind** of order ν and $-\nu$, respectively and converge at least on $(0, \infty)$.

If ν is not an integer, then the general solution to Bessel's equation is $y = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$.

Definition: The function $Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$ (ν a non-integer) is a **Bessel function of the second kind**. It can be shown to be linearly independent with $J_\nu(x)$.

If ν is an integer, then it can be shown that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly dependent. However, $J_\nu(x)$ and $Y_\nu(x)$ are independent. As such, for an integer value of ν , we use $J_\nu(x)$ and $Y_\nu(x)$ (in this case, $Y_\nu(x)$ is actually defined by a limiting process as $\nu \rightarrow m$, where m is an integer). In this case we can give the general solution to Bessel's equation as $y = c_1 J_\nu(x) + c_2 Y_\nu(x)$.

Example: Find the general solution of the given differential equation on $(0, \infty)$.

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

$\nu = \pm 1$ (integers)

$$y = c_1 J_1(x) + c_2 Y_1(x)$$

Alternate forms of Bessel's equation:

Using a substitution we can find that the general solution to the DE $x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0$ is $y = c_1J_\nu(\alpha x) + c_2Y_\nu(\alpha x)$.

Definition: The preceding differential equation is called the **parametric Bessel equation of order ν** .

Definition: The differential equation $x^2y'' + xy' - (x^2 + \nu^2)y = 0$ is the **modified Bessel equation of order ν** . It can be converted to a Bessel equation by the substitution $t = ix$, where $i^2 = -1$; the result is...

Definition: The **modified Bessel function of the first kind** of order ν , $I_\nu(x) = i^{-\nu}J_\nu(ix)$.

The general solution to the modified Bessel equation of order ν (ν not an integer) is $y = c_1I_\nu(x) + c_2I_{-\nu}(x)$.

$\begin{matrix} \text{J - non int} \\ \text{Y - int} \end{matrix} \quad \begin{matrix} \text{I - non int} \\ \text{K - int} \end{matrix}$

Definition: As before, for non-integer values of ν we define the **modified Bessel function of the second kind**, $K_\nu(x) = \frac{\pi I_{-\nu}(x) - I_\nu(x)}{2 \sin \nu\pi}$ and extend this definition to integer values using a limit process similar to that referenced when defining $Y_\nu(x)$.

And as before, the general solution to the modified Bessel equation of order ν (for ν an integer) is $y = c_1I_\nu(x) + c_2K_\nu(x)$. A parametric form of the modified Bessel equation and associated solution exists such as we saw above.

Definition: An equation of the form $x^2y'' + xy' - (\alpha^2x^2 + \nu^2)y = 0$ is a **parametric form of the modified Bessel equation of order ν** and can be obtained from a modified Bessel equation using a change of variables.

The general solution for such an equation is $y = c_1I_\nu(\alpha x) + c_2K_\nu(\alpha x)$.

Example: Find the general solution of the given differential equation on $(0, \infty)$.

$$x^2 y'' + xy' - (2x^2 + 64)y = 0$$

$$\alpha^2 = 2 \Rightarrow \alpha = \sqrt{2}, \quad \nu = \pm 8i$$

$$y = c_1 I_8(\sqrt{2}x) + c_2 K_8(\sqrt{2}x)$$

Example: The general solution to $y'' + \frac{1-2a}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2-p^2c^2}{x^2}\right)y = 0$,

$p \geq 0$ is $y = x^a [c_1 J_p(bx^c) + c_2 Y_p(bx^c)]$. Use this to find the general solution of the given differential equation on $(0, \infty)$.

$$xy'' + 3y' + xy = 0$$

$$y'' + \frac{3}{x}y' + y = 0$$

$$1-2a=3 \Rightarrow a=-1$$

$$b^2c^2x^{2c-2} + \frac{a^2-p^2c^2}{x^2} = 1x^0$$

$$2c-2=0 \Rightarrow c=1$$

$$1=p^2c^2 \Rightarrow p=1$$

$$b^2c^2=1 \Rightarrow b=1$$

$$y = x^{-1} [c_1 J_1(x) + c_2 Y_1(x)]$$

Legendre polynomials:

Definition: The differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

is Legendre's equation of order n .

Note that $x = 0$ is an ordinary point. Starting with $y = \sum_{k=0}^{\infty} c_k x^k, \dots$

$$\text{leads to } 2c_2 + 6c_3x - 2c_1x + n(n+1)c_0 + n(n+1)c_1x + \sum_{j=2}^{\infty} \left\{ (j+2)(j+1)c_{j+2} + [-j(j-1) - 2j + n(n+1)]c_j \right\} x^j = 0$$

$$\text{We find that } c_{j+2} = -\frac{(n-j)(n+j+1)}{(j+2)(j+1)}c_j, j = 2, 3, 4, \dots$$

$$c_2 = -\frac{n(n+1)}{2}c_0$$

$$c_4 = -\frac{(n-2)(n+3)}{4 \cdot 3}c_2 = \frac{(n-2)n(n+1)(n+3)}{4!}c_0$$

This terminates if n is even.

$$c_3 = -\frac{(n-1)(n+2)}{6}c_1$$

$$c_5 = -\frac{(n+4)(n-3)}{5 \cdot 4}c_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1$$

This terminates if n is odd.

Traditional values for c_i are

$$\text{For } n = 0, c_0 = 1; \text{ for } n = 2, 4, 6, \dots, c_0 = (-1)^{n/2} \frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n}$$

$$\text{For } n = 1, c_1 = 1; \text{ for } n = 3, 5, 7, \dots, c_1 = (-1)^{(n-1)/2} \frac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)}$$

Definition: For each value of n the result is a **Legendre polynomial of order n** .

The first six Legendre polynomials are

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_1(x) = x$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$